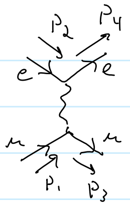
Note: No contribution from

Consider $e + \mu \rightarrow e + \mu$



$$\Rightarrow \quad \mathcal{M} = \bar{u}(3)\, i g_e \gamma^\mu u(1)\, \bar{u}(4)\, i g_e \gamma^\nu u(2) \left(\frac{-i\, \eta_{\mu\nu}}{(p_4 - p_2)^2} \right) i$$

Once we have $\mathcal{M}$, we can evaluate it given the following:

$u(i) = \alpha_i u^{(1)}(p_i) + \beta_i u^{(2)}(p_i) \qquad i = 1, 2, 3, 4$

However in most experiments we average over $\alpha_1, \beta_1, \alpha_2, \beta_2$ and sum over $\alpha_3, \beta_3, \alpha_4, \beta_4$.

To do the avg/sum we first note that $\mathcal{M}$ contains terms of the form:

$$\mathcal{M} = \cdots \times \underbrace{\bar{u}(a)\, \Gamma_1\, u(b)}_{\text{spinor sandwhich}}$$

Γ: some combination of spin matrices

1: label to distinguish from other cases

Then:

$$|\mathcal{M}|^2 = \cdots \bar{u}(a) \Gamma_1 u(b) \underbrace{\left[\bar{u}(a) \Gamma_2 u(b) \right]^*}_{\substack{\text{This thing is a number in spin space} \\ \text{so we can freely transpose it to form } \dagger.}} \cdots$$

Note: We give this a different label. In particular if Γ_1 carries spacetime indices μ, ν then Γ_2 will need different indices α, λ to avoid confusing contractions.

$$= \cdots \bar{u}(a) \Gamma_1 u(b) \underbrace{\left[\bar{u}(a) \Gamma_2 u(b) \right]^\dagger}$$

$$u(b)^\dagger \Gamma_2^\dagger \bar{u}(a)^\dagger = u(b)^\dagger \Gamma_2^\dagger \left(u(a)^\dagger \gamma^0 \right)^\dagger$$

$$= u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)$$

insert $\gamma^0 \gamma^0 = I$ and use $\gamma^0 = \gamma^{0\dagger}$

$$= u(b)^\dagger \gamma^0 \gamma^0 \Gamma_2^\dagger \gamma^0 u(a)$$

$$= \bar{u}(b)\, \bar{\Gamma}_2\, u(a)$$

$\bar{\Gamma}_2 = \gamma^0 \Gamma_2^\dagger \gamma^0$

$$= \cdots \bar{u}(a) \Gamma_1 u(b) \bar{u}(b) \bar{\Gamma}_2 u(a)$$

Now to avg/sum the first step will be to sum so we can use $\sum_s u^{(s)} \bar{u}^{(s)} = \gamma^\mu p_\mu + mc = \not{p} + mc$

$$\sum_{s_b} |\mathcal{M}|^2 = \cdots \underbrace{\bar{u}(a)}_{\text{Row}}\, \underbrace{\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2}_{\text{Matrix}}\, \underbrace{u(a)}_{\text{Column}} \cdots$$

Now we can use that $RMC = \# = \mathrm{Tr}(MCR)$

$$\sum_{s_b} |\mathcal{M}|^2 = \cdots \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2\, u(a) \bar{u}(a) \right] \cdots$$

Now sum over s_a to get:

$$\sum_{s_a, s_b} |\mathcal{M}|^2 = \cdots \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2 (\not{p}_a + m_a c) \right] \cdots$$

If at some point we had summed over $v\bar{v} \Rightarrow \not{p} - mc$.

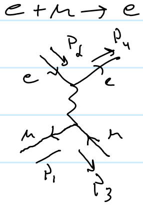
Note: There are no spinors left in the expression! Only p_μ's and γs!

$$\boxed{\text{To impose sum over } s_a, s_b \text{ replace } \bar{u}(a)\Gamma_1 u(b)\left[\bar{u}(a)\Gamma_2 u(b)\right]^* \Rightarrow Tr\left[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)\right]}$$

Let's put this result to work:

$e + \mu^+ \to e + \mu^+$

$$M = -\frac{g_e^2}{(p_4-p_2)^2}\,\bar{v}(1)\gamma^\mu v(3)\,\bar{u}(4)\gamma^\nu u(2)\,g_{\mu\nu}$$



$$|M|^2 = \frac{g_e^4}{(p_4-p_2)^4}\,\bar{v}(1)\gamma^\mu v(3)\,\bar{u}(4)\gamma^\nu u(2)\,g_{\mu\nu}\left[\bar{v}(1)\gamma^\lambda v(3)\,\bar{u}(4)\gamma^\alpha u(2)\,g_{\lambda\alpha}\right]^*$$

$$Tr\left[\gamma^\mu(\not{p}_3 - m_3 c)\bar{\gamma}^\lambda(\not{p}_1 - m_1 c)\right]$$

$$Tr\left[\gamma^\nu(\not{p}_2 + m_2 c)\bar{\gamma}^\alpha(\not{p}_4 + m_4 c)\right]$$

2 incoming particles w/ 2 spin states each

$$\langle |M|^2 \rangle = \frac{1}{4}\frac{g_e^4}{(p_4-p_2)^4}\,Tr\left[\gamma^\mu(\not{p}_3 - m_3 c)\bar{\gamma}^\lambda(\not{p}_1 - m_1 c)\right]Tr\left[\gamma^\nu(\not{p}_2 + m_2 c)\bar{\gamma}^\alpha(\not{p}_4 + m_4 c)\right]g_{\mu\nu}g_{\lambda\alpha}$$

To continue we need to know how to evaluate traces in spin space. Fortunately there are some useful results:

a) $Tr(\gamma^\mu \gamma^\alpha \gamma^\lambda \gamma^\beta) = 4(\eta^{\mu\alpha}\eta^{\lambda\beta} - \eta^{\mu\lambda}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\lambda})$

b) $Tr(\gamma^\mu \gamma^\alpha \gamma^\lambda) = 0$

c) $Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$ $\qquad \gamma^0 \gamma^{\lambda\dagger} \gamma^0 = \gamma^\lambda$ (nontrivial but true!)

Then: $Tr[\gamma^\mu(\not{p}_a \pm m_a c)\bar{\gamma}^\lambda(\not{p}_b \pm m_b c)] = X_\pm$

Using: $Tr[A+B] = Tr A + Tr B$

We have: $X_\pm = Tr(\gamma^\mu \not{p}_a \gamma^\lambda \not{p}_b) \pm Tr(\gamma^\mu \not{p}_a \gamma^\lambda m_b c) \pm Tr(\gamma^\mu m_a c \gamma^\lambda \not{p}_b) + Tr(\gamma^\mu m_a c \gamma^\lambda m_b c)$

Using: $Tr[\alpha M] = \alpha Tr M$

α scalar in space where M is a matrix!

$$X_\pm = p_{a\alpha} p_{b\beta} \underbrace{Tr(\gamma^\mu \gamma^\alpha \gamma^\lambda \gamma^\beta)}_{4(\eta^{\mu\alpha}\eta^{\lambda\beta} - \eta^{\mu\lambda}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\lambda})} \pm m_b c\, p_{a\alpha} \underbrace{Tr(\gamma^\mu \gamma^\alpha \gamma^\lambda)}_{0} \pm m_a c\, p_{b\alpha} \underbrace{Tr(\gamma^\mu \gamma^\lambda \gamma^\alpha)}_{0} + (m_a c)(m_b c) \underbrace{Tr(\gamma^\mu \gamma^\lambda)}_{4\eta^{\mu\lambda}}$$

$$= 4(p_a^\mu p_b^\lambda - \eta^{\mu\lambda} p_a \cdot p_b + p_a^\lambda p_b^\mu) + 4 m_a m_b c^2 \eta^{\mu\lambda} \Rightarrow X_+ = X_-$$

Then for our $e + \mu \to e + \mu$ result:

$$\langle |M|^2 \rangle = \frac{1}{4} \frac{g_e^4}{(p_4 - p_2)^4} 16 \, (p_3^\mu p_1^\lambda - \eta^{\mu\lambda} p_3 \cdot p_1 + p_3^\lambda p_1^\mu + m_\mu^2 c^2 \eta^{\mu\lambda})(p_2^\nu p_4^\alpha - \eta^{\nu\alpha} p_2 \cdot p_4 + p_2^\alpha p_4^\nu + m_e^2 c^2 \eta^{\nu\alpha}) \eta_{\mu\nu} \eta_{\lambda\alpha}$$

$$= \frac{8 g_e^4}{(p_4 - p_2)^4} \left[(p_3 \cdot p_2)(p_1 \cdot p_4) + (p_3 \cdot p_4)(p_2 \cdot p_1) - (p_3 \cdot p_1) m_e^2 c^2 - (p_2 \cdot p_4) m_\mu^2 c^2 + 2 m_e^2 m_\mu^2 c^4 \right]$$

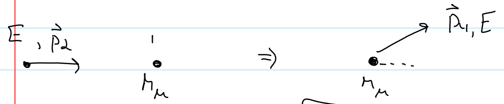
Note: Our final expression is in terms of only 4momenta, i.e. E and $\vec{p}$!

Recall that for 2-body scattering: $\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}$ for the CM-frame

If we then take the approximation $m_\mu >> m_e$ and assume that $E_2 = E_e << m_\mu c^2$ we find:

$$|\vec{p}_f| = |\vec{p}_i| \quad , \quad E_1 + E_2 = \underbrace{m_\mu c^2 + E_2}_{\text{CM-frame is essentially the rest frame of } \mu} \simeq m_\mu c^2$$

Then: $\left\langle\frac{d\sigma}{d\Omega}\right\rangle = \left(\frac{\hbar}{8\pi m_\mu c}\right)^2 \langle |M|^2\rangle$



$|\vec{p}_2| = |\vec{p}_4| \equiv |\vec{p}|$

$$p_1 = (m_\mu c, 0) \quad p_2 = \left(\frac{E}{c}, \vec{p}_2\right) \quad p_3 = (m_\mu c, \vec{0}) \quad p_4 = \left(\frac{E}{c}, \vec{p}_4\right)$$

approximate since $m_\mu >> m_e$

Then: $(p_4 - p_2)^2 = (0, \vec{p}_4 - \vec{p}_2)^2 = 0 - (\vec{p}_4 - \vec{p}_2)^2 = -\vec{p}_4^{\,2} - \vec{p}_2^{\,2} + 2\vec{p}_2\cdot\vec{p}_4$

$= -2\vec{p}^{\,2}(1-\cos\theta)$

$= -4\vec{p}^{\,2}\sin^2\frac{\theta}{2}$

Using $\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$ and $\cos 0 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}$

$\Rightarrow (1-\cos\theta) = 2\sin^2\frac{\theta}{2}$

$p_2\cdot p_4 = \left(\frac{E}{c}\right)^2 - \vec{p}_2\cdot\vec{p}_4 = m_e^2c^2 + 2\vec{p}^{\,2}\sin^2\left(\frac{\theta}{2}\right) \longleftarrow \left(\frac{E}{c}\right)^2 - p^2\cos\theta = \left(\frac{E}{c}\right)^2 - p^2 + p^2 - p^2\cos\theta$

$= m_e^2c^2 + p^2(1-\cos\theta)$

$= m_e^2c^2 + 2p^2\sin^2\frac{\theta}{2}$

$p_1\cdot p_3 = m_\mu^2 c^2$

$(p_1\cdot p_2)(p_3\cdot p_4) = (p_1\cdot p_4)(p_2\cdot p_3) = m_\mu^2 E^2$

$$\langle|M|^2\rangle = \frac{8 g_e^4}{16 p^4\sin^4\frac{\theta}{2}}\left[2m_\mu^2E^2 - m_\mu^2c^2m_e^2c^2 - \left(m_e^2c^2 + 2p^2\sin^2\frac{\theta}{2}\right)m_\mu^2c^2 + 2m_e^2m_\mu^2c^4\right] = \frac{1}{2}\left(\frac{g_e^2}{p^2\sin^2\frac{\theta}{2}}\right)^2\left[2m_\mu^2E^2 - 2m_\mu^2c^2p^2\sin^2\frac{\theta}{2}\right]$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_\mu c}\right)^2\left(\frac{g_e^2}{p^2\sin^2\frac{\theta}{2}}\right)^2 m_\mu^2c^2\left[\frac{E^2}{c^2} - p^2\sin^2\frac{\theta}{2}\right] = \left(\frac{\hbar}{8\pi}\right)^2\left(\frac{g_e^2}{p^2\sin^2\frac{\theta}{2}}\right)^2\left[m_e^2c^2 + p^2 - p^2\sin^2\frac{\theta}{2}\right]$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{g_e^2\hbar}{8\pi p^2\sin^2\left(\frac{\theta}{2}\right)}\right)^2\left[(m_ec)^2 + \vec{p}^{\,2}\cos^2\left(\frac{\theta}{2}\right)\right]$$

Mott Formula

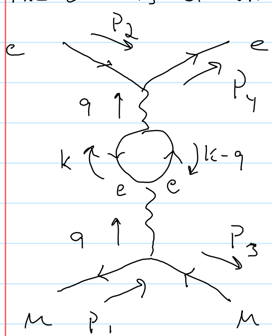
Note: $g_e = e\sqrt{\frac{4\pi}{\hbar c}} = \sqrt{4\pi\alpha}$

In the non-relativistic limit $\vec{p}^{\,2} << (m_ec)^2$ this becomes:

$$\frac{d\sigma}{d\Omega} = \left(\frac{g_e^2\hbar m_e c}{8\pi m_e^2v^2\sin^2\left(\frac{\theta}{2}\right)}\right)^2 = \left(\frac{e^2 4\pi\hbar m_e c}{8\pi\hbar c\, m_e^2v^2\sin^2\left(\frac{\theta}{2}\right)}\right)^2 = \left(\frac{e^2}{8m_e v^2\sin^2\left(\frac{\theta}{2}\right)}\right)^2$$

Rutherford Formula

The effects of virtual particle pairs start at 4th order with the largest contribution being:



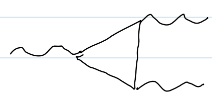
To evaluate a diagram like this with a purely internal loop of matter we need a new Feynman rule: For internal loops of matter write down the ordered product of vertex factors and propagators, the take the trace (in spin space) and × (−1)

$$\Rightarrow \mathcal{M}_{1\text{-loop}} = -\frac{i g_e^4}{q^4}\left[\bar{v}(1)\gamma^\mu v(3)\right] \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu(\not{k} + m_e c)\gamma_\nu(\not{k} - \not{q} + m_e c)\right]}{(k^2 - m_e^2 c^2)\left[(k-q)^2 - m_e^2 c^2\right]}}_{\text{This contribution is actually divergent and will eventually lead us to the topic of renormalization.}} \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

$\boxed{q \equiv p_2 - p_4}$

We also get another "rule of thumb" associated w/ the new Feynman rule.

Furry's Theorem: When constructing diagrams you can ignore contributions from closed internal matter loops w/ an odd # of vertices (since $\mathcal{M} = 0$).

An application is photon "decay":  $\Rightarrow \mathcal{M} = 0$